Augmented Matrices and Row Echelon Form

Definition: Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is a $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$.

We call the matrix A the corresponding coefficient matrix and the $m \times (n+1)$ matrix $[A|\mathbf{b}]$ the corresponding augmented matrix.

Example 1: Write the corresponding coefficient matrix and the corresponding augmented matrix.

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad [A|\vec{b}] = \left[\begin{array}{cc|c} 2 & -1 & -2 \\ 2 & 1 & 1 \\ 0 & -2 & 3 \end{array}\right]$$

Example 2: Write the linear system $A\mathbf{x} = \mathbf{b}$ that has the corresponding augmented matrix given in *matrix-vector form*, *vector form*, and *equation form*. Use *back-substitution* to solve the linear system.

$$\begin{array}{c} \qquad A \qquad\quad \vec{b} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right] \end{array} \tag{1}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$ (matrix-vector form)

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$ (vector form)

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_2 + x_3 &= 0 \\ -x_3 &= 2 \end{aligned}$$

The linear system has the unique solution $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$x_3 = -2$

$2x_2 = -x_3 = 2 \Rightarrow x_2 = 1$

$x_1 = 1 - x_2 - x_3 = 1 - 1 - (-2) = 2$

Note 1: The linear system in example 2 could be solved using back-substitution because the corresponding augmented matrix $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right]$ is in *row echelon form.*

1.) ✓ 2.) ✓

> *Definition:* A matrix is said to be in <u>row echelon form</u> if
>
> 1. Any rows consisting entirely of zeroes are at the bottom.
> 2. In each nonzero row, the first nonzero entry (called the <u>leading entry</u>) is in a column to the left of any leading entries below it.

Example 3: Which matrices (if any) are in row echelon form. If the matrix is not in row echelon form explain why not.

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix}$$

1.) X not in form

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1.) ✓
2.) ✓ Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

1.) ✓
2.) X no

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

1.) ✓ 2.) ✓

Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

1.) ✓
2.) X no

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.) ✓
2.) ✓ Yes